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Mode conversion in an imperfect waveguide

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Abstract. A formalism is described for calculating the mode conversion induced in a straight waveguide by imperfections which break its translational symmetry, and in one whose direction changes along its length. The method employs 'quasimodes' which are adapted to the local nature of the guide. The method is particularly useful when the spatial frequency of the imperfections is small compared with the differences in wavenumbers of the guided modes involved or when the radius of a bend is much greater than the width of the guide.

1. Introduction

It is well known that straight dielectric and metal waveguides whose properties are constant along their length support spectra of propagating, radiation and evanescent electromagnetic modes. Any imperfection in a guide which breaks its translational symmetry will change the properties of its modes. It is often convenient to think of the imperfection as a perturbation which converts power between modes of the perfect guide.

The quantum-mechanical analogy of a potential varying in time (eg Schiff 1968, p 280) has been persued by Marcuse (1969) to develop a perturbation formalism for calculating the extent of mode conversion in dielectric waveguides. Such a theory imposes no condition on the spatial frequency of the perturbation, but requires it to be of small amplitude. It is, indeed, a perturbation theory, as it assumes that the real guide does not differ greatly from the perfect one.

Snyder (1970, 1971) has developed an alternative method for calculating the mode conversion in a straight guide whose properties (cross section, dielectric constant and relative permeability) change along its length. We have developed independently a formalism which is mathematically equivalent to Snyder's but whose derivation shows the close analogy between the method and that of the adiabatic approximation (Schiff 1968, p 289) used in quantum mechanics to treat a potential which varies in time. Our paper describes this method paying full attention to problems of completeness, orthogonality and the unity of the quasimodes which arise in the development.

The interest in this adiabatic procedure arises from the possibility of using optical fibres as waveguides for communication systems. In such guides the radiation wavelength is invariably small compared with the spatial range of the imperfections which themselves may be large in amplitude. Analogously the adiabatic approximation is known to be useful in quantum mechanics for an arbitrarily large change in potential provided the change takes place slowly on a time scale determined by the frequency differences of energy eigenstates.

We also present an adiabatic method for treating mode conversion in a bending waveguide. This problem has also been considered by Bahar (1969). In this case there is no alternative perturbation treatment since a bent guide can in no satisfactory way be considered a small perturbation on a straight one.

The method which we use and the physical arguments behind it are as follows.

The problem is to solve Maxwell's equations in the vicinity of the slowly varying bending waveguide. In general this is difficult because of the lack of translational symmetry. To re-introduce this symmetry we construct at each point z measured along the guide an imaginary, *straight* guide whose properties are constant along its length and coincide with those of the real guide at z (figures 1 and 2).

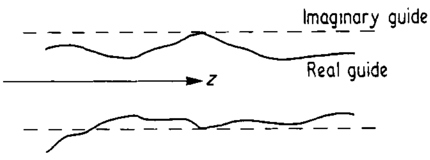

Figure 1. Slowly varying linear waveguide.

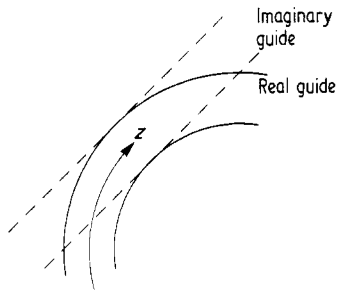

Figure 2. Slowly bending waveguide.

We can solve Maxwell's equations for these imaginary guides since they are straight and uniform. From these solutions we construct the quasimodes, sets of electromagnetic fields which satisfy the appropriate boundary conditions at each point z along the guide. These quasimodes are not eigenmodes of the real guide but they form a complete set and are 'almost' eigenmodes if the variations are slow or the bends are gentle. We now expand the true fields in terms of these quasimodes and by forcing this expansion to satisfy Maxwell's equations we can determine its coefficients. In this way we solve the problem of quasimode conversion and hence of mode conversion since as the guide becomes straight and constant the quasimodes collapse into eigenmodes.

In § 2.1 quasimodes for an imperfect linear guide are described and their properties discussed. Section 2.2 considers the expansion of the solution in terms of these quasimodes and a formal solution is derived in § 2.3 equivalent to that obtained by Snyder (1970). Section 3.1 solves the problem of the bending waveguide of which a simple example is illustrated in § 3.2.

Application to the more complicated structure of optical fibres will be the subject of later work.

2. The imperfect straight guide

2.1. *Quasimode formalism*

The electromagnetic field in the vicinity of a waveguide must satisfy Maxwell's equations:

$$\nabla . \boldsymbol{D} = \nabla . \epsilon \boldsymbol{E} = 0 \tag{1}$$

$$\nabla . \boldsymbol{B} = \nabla . \mu \boldsymbol{H} = 0 \tag{2}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = \frac{\mathrm{i}\omega\mu}{c}\boldsymbol{H} \tag{3}$$

$$\nabla \times \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} = -\frac{\mathrm{i}\omega\epsilon}{c}\boldsymbol{E} \tag{4}$$

where we allow ϵ and μ to vary in space and we have assumed a time variation proportional to $e^{-i\omega t}$. Equations (1) and (2) follow automatically from (3) and (4), but are included here because they are used explicitly later on. Gaussian units are used throughout this paper.

We consider a system of coordinates with guide axis in the z direction $\boldsymbol{\varepsilon}_3$. The choice of axis may be somewhat arbitrary for an imperfect guide. Since the solution must be independent of this choice we suppose it made for convenience. We imagine constructed a perfect guide whose properties throughout its length are those of the real guide at z. If we call z' the coordinate along this imaginary guide then its eigenmodes will be solutions of equations (1)–(4) with ∇ replaced by

$$\nabla' \equiv \boldsymbol{\varepsilon}_1\frac{\partial}{\partial x} + \boldsymbol{\varepsilon}_2\frac{\partial}{\partial y} + \boldsymbol{\varepsilon}_3\frac{\partial}{\partial z'}.$$

Because the imaginary guide is perfect the z' dependence of the modes will be proportional to $\exp(\pm ik_n(z)z')$ for forward or backward waves, where $k_n(z)$ is the wavenumber, depending on z through the cross section of the real guide chosen as a template for the imaginary one. The x, y dependence of the forward/backward $(\pm)$ modes will satisfy the equations

$$(\nabla_t \pm \mathrm{i}k_n(z)\boldsymbol{\varepsilon}_3) \cdot \epsilon \boldsymbol{E}_n^{\pm}(x, y; z) = 0 \tag{5}$$

$$(\nabla_t \pm \mathrm{i}k_n(z)\boldsymbol{\varepsilon}_3) \cdot \mu \boldsymbol{H}_n^{\pm}(x, y; z) = 0 \tag{6}$$

$$(\nabla_t \pm \mathrm{i}k_n(z)\boldsymbol{\varepsilon}_3) \times \boldsymbol{E}_n^{\pm}(x, y; z) = \frac{\mathrm{i}\omega\mu}{c}\boldsymbol{H}_n^{\pm}(x, y; z) \tag{7}$$

$$(\nabla_t \pm \mathrm{i}k_n(z)\boldsymbol{\varepsilon}_3) \times \boldsymbol{H}_n^{\pm}(x, y; z) = -\frac{\mathrm{i}\omega\epsilon}{c}\boldsymbol{E}_n^{\pm}(x, y; z) \tag{8}$$

where

$$\nabla_t \equiv \boldsymbol{\varepsilon}_1\frac{\partial}{\partial x} + \boldsymbol{\varepsilon}_2\frac{\partial}{\partial y}.$$

The z dependence of $\boldsymbol{E}_n^{\pm}$, $\boldsymbol{H}_n^{\pm}$ and k_n will be slow compared with the z' dependence of $\exp(\pm ik_n(z)z')$ since the former originates from the slow imperfections.

From these equations for $\boldsymbol{E}_n^{\pm}$ and $\boldsymbol{H}_n^{\pm}$ we can derive (appendix 1) the wave equations obeyed by their transverse components:

$$\left(\begin{pmatrix}\mu\\ \epsilon\end{pmatrix}\nabla_t \times \begin{pmatrix}\mu^{-1}\\ \epsilon^{-1}\end{pmatrix}\nabla_t \times - \nabla_t\begin{pmatrix}\epsilon^{-1}\\ \mu^{-1}\end{pmatrix}\nabla_t \cdot \begin{pmatrix}\epsilon\\ \mu\end{pmatrix} - \frac{\omega^2\mu\epsilon}{c^2} + k_n^2\right)\begin{pmatrix}\boldsymbol{E}_{nt}^{\pm}(x, y; z)\\ \boldsymbol{H}_{nt}^{\pm}(x, y; z)\end{pmatrix} = 0. \tag{9}$$

These equations, together with the boundary conditions, constitute an eigenvalue equation for the transverse fields and wavenumbers. The longitudinal components follow from the transverse components when we consider equations (7) and (8) resolved along ε_3:

$$\frac{i\omega\mu}{c}\varepsilon_3 H^{\pm}_{nz}(x, y; z) = \nabla_t \times \boldsymbol{E}^{\pm}_{nt}(x, y; z) \tag{10}$$

$$-\frac{i\omega\epsilon}{c}\varepsilon_3 E^{\pm}_{nz}(x, y; z) = \nabla_t \times \boldsymbol{H}^{\pm}_{nt}(x, y; z). \tag{11}$$

When ϵ and μ are piecewise independent of x and y with discontinuities at boundaries then equations (9) become

$$\left(\nabla_t^2 + \frac{\omega^2\mu\epsilon}{c^2} - k_n^2(z)\right)\begin{pmatrix}\boldsymbol{E}^{\pm}_{nt}(x, y; z)\\ \boldsymbol{H}^{\pm}_{nt}(x, y; z)\end{pmatrix} = 0 \tag{12}$$

and the variations of ϵ and μ appear in the boundary conditions.

We shall continue to define $\boldsymbol{E}^{\pm}_n$ and $\boldsymbol{H}^{\pm}_n$ via equations (9)–(11). These are solutions of an eigenvalue problem and so form a complete set in x, y; hence we can express any field generated by a source in the vicinity of the guide as a linear combination of these solutions. Their orthonormality properties are discussed in appendix 2. It is found that

$$\int \boldsymbol{H}^{\pm}_{mt} \times \boldsymbol{E}^{\pm}_{nt} \cdot \mathrm{d}\boldsymbol{S} = \text{constant} \times \delta_{nm} \tag{13}$$

where the constant depends on the normalization and the Kronecker delta symbol is replaced by a Dirac delta function for continuum modes. The integral is over the area of the plane perpendicular to the axis.

The quasimodes of the real guide are constructed as follows:

$$\begin{pmatrix}\hat{\boldsymbol{E}}^{\pm}_n(\boldsymbol{r}, t)\\ \hat{\boldsymbol{H}}^{\pm}_n(\boldsymbol{r}, t)\end{pmatrix} = \begin{pmatrix}\boldsymbol{E}^{\pm}_n(x, y; z)\\ \boldsymbol{H}^{\pm}_n(x, y; z)\end{pmatrix} \exp\left(\pm i \int^z k_n(z')\,\mathrm{d}z' - i\omega t\right). \tag{14}$$

The fields are those appropriate to the form of the guide at z; the phase is the total of the increments previously acquired along the guide.

These quasimodes satisfy:

(i) the boundary conditions at any point on the real guide;

(ii) Maxwell's equations, provided we ignore all z dependences except those in the exponents.

Clearly as the real guide becomes uniform its quasimodes collapse into its eigenmodes.

2.2. The expansion

The fields are expanded as

$$\begin{pmatrix}\boldsymbol{E}(\boldsymbol{r}, t)\\ \boldsymbol{H}(\boldsymbol{r}, t)\end{pmatrix} = \sum_n A^{+}_n(z)\begin{pmatrix}\hat{\boldsymbol{E}}^{+}_n(\boldsymbol{r}, t)\\ \hat{\boldsymbol{H}}^{+}_n(\boldsymbol{r}, t)\end{pmatrix} + \sum_n A^{-}_n(z)\begin{pmatrix}\hat{\boldsymbol{E}}^{-}_n(\boldsymbol{r}, t)\\ \hat{\boldsymbol{H}}^{-}_n(\boldsymbol{r}, t)\end{pmatrix} \tag{15}$$

where Σ_n represents a sum over all discrete modes and an integral over any continuum modes, and the $A^{\pm}_n(z)$ are to be determined.

The assumption that this expansion is permissible, that the $A^{\pm}_n(z)$ are not over- or under-determined, needs justifying.

To do this we observe that since the eigenfunctions $\boldsymbol{E}_{nt}^{\pm}(x, y; z)$ of (9) belong to the same eigenvalue, $k_n^2(z)$, we are at liberty to chose

$$\boldsymbol{E}_{nt}^{+}(x, y; z) = \boldsymbol{E}_{nt}^{-}(x, y; z) \equiv \boldsymbol{E}_{nt}(x, y; z), \qquad \text{say.} \tag{16a}$$

For the same reason $\boldsymbol{H}_{nt}^{\pm}$ are multiples of one another and having chosen (16*a*) it follows from equations (5), (6), (10) and (11) that

$$\boldsymbol{H}_{nt}^{+}(x, y; z) = -\boldsymbol{H}_{nt}^{-}(x, y; z) \equiv \boldsymbol{H}_{nt}(x, y; z), \qquad \text{say.} \tag{16b}$$

By writing the transverse components of the expansion (15) in terms of $\boldsymbol{E}_{nt}$ and $\boldsymbol{H}_{nt}$ and bearing in mind the completeness properties of these transverse fields, it is clear that the arbitrariness is just necessary and sufficient. We can now write the orthogonality relation (13) as

$$\int \boldsymbol{H}_{mt} \times \boldsymbol{E}_{nt} \cdot \mathrm{d}\boldsymbol{S} = \text{constant} \times \delta_{nm}. \tag{13a}$$

To see that the longitudinal component of the expansion (15) is also correct we merely have to operate on its transverse components with $\nabla_t \times$ and use the relations (1), (2), (10), and (11) from which the z components of (15) follow immediately.

We have thus been able to show that it is permissible to expand the electromagnetic field of the guide in terms of the quasimodes.

2.3. *Formal solution*

We have now to determine the coefficients $A_n^{\pm}(z)$ by making the expansions (15) satisfy Maxwell's equations. Substituting for $\boldsymbol{E}$ and $\boldsymbol{H}$ from equations (15) on the left of (3) and (4) we find

$$\sum_n \frac{\partial}{\partial z}\left\{A_n^{+}(z)\begin{pmatrix}\boldsymbol{E}_{nt}^{+}\\ \boldsymbol{H}_{nt}^{+}\end{pmatrix}\right\}\exp\left(+\mathrm{i}\int^z k_n(z')\,\mathrm{d}z'\right) + \sum_n \frac{\partial}{\partial z}\left\{A_n^{-}(z)\begin{pmatrix}\boldsymbol{E}_{nt}^{-}\\ \boldsymbol{H}_{nt}^{-}\end{pmatrix}\right\}\exp\left(-\mathrm{i}\int^z k_n(z')\,\mathrm{d}z'\right) = 0. \tag{17a}$$

By operating on (17*a*) with ∇_t and simplifying with (10) and (11) we can find relations involving the longitudinal components:

$$\sum_n \frac{\partial}{\partial z}\left\{A_n^{+}(z)\begin{pmatrix}\epsilon E_{nz}^{+}\\ \mu H_{nz}^{+}\end{pmatrix}\right\}\exp\left(+\mathrm{i}\int^z k_n(z')\,\mathrm{d}z'\right) + \sum_n \frac{\partial}{\partial z}\left\{A_n^{-}(z)\begin{pmatrix}\epsilon E_{nz}^{-}\\ \mu H_{nz}^{-}\end{pmatrix}\right\}\exp\left(-\mathrm{i}\int^z k_n(z')\,\mathrm{d}z'\right) = 0. \tag{17b}$$

Equations (17*b*) could also have been derived directly from the expansions (15) and Maxwell's equations (1) and (2).

It remains to extract explicit expressions for the $A_n^{\pm}(z)$. Equations (17) constitute six sets of scalar equations which (consistently) over-determine the two sets of quantities $A_n^{\pm}(z)$. Since in some cases the axial components of $\boldsymbol{E}$ or $\boldsymbol{H}$ vanish so that their associated equations are satisfied identically, we choose to deal with the equations involving the transverse fields. It is easily shown that these equations are always nontrivial. The transverse equations can be simplified with our convention (16) to read

$$\sum_n \frac{\partial}{\partial z}(A_n^+ \boldsymbol{E}_{nt}) \exp\left(+\mathrm{i} \int^z k_n(z')\,\mathrm{d}z' \right) + \sum_n \frac{\partial}{\partial z}(A_n^- \boldsymbol{E}_{nt}) \exp\left(-\mathrm{i} \int^z k_n(z')\,\mathrm{d}z' \right) = 0 \tag{18}$$

$$\sum_n \frac{\partial}{\partial z}(A_n^+ \boldsymbol{H}_{nt}) \exp\left(+\mathrm{i} \int^z k_n(z')\,\mathrm{d}z' \right) - \sum_n \frac{\partial}{\partial z}(A_n^- \boldsymbol{H}_{nt}) \exp\left(-\mathrm{i} \int^z k_n(z')\,\mathrm{d}z' \right) = 0. \tag{19}$$

Operating on (18) and (19) with $\boldsymbol{H}_{mt} \times$ and $\boldsymbol{E}_{mt} \times$ respectively, assuming them normalized correctly and using (13*a*) we find

$$\begin{aligned}
&\frac{\partial A_m^+}{\partial z} \exp\left(+\mathrm{i} \int^z k_m(z')\,\mathrm{d}z' \right) + \frac{\partial A_m^-}{\partial z} \exp\left(-\mathrm{i} \int^z k_m(z')\,\mathrm{d}z' \right) \\
&\qquad = -\sum_n A_n^+ \int \boldsymbol{H}_{mt} \times \frac{\partial \boldsymbol{E}_{nt}}{\partial z} \cdot \mathrm{d}\boldsymbol{S} \exp\left(+\mathrm{i} \int^z k_n(z')\,\mathrm{d}z' \right) \\
&\qquad\quad - \sum_n A_n^- \int \boldsymbol{H}_{mt} \times \frac{\partial \boldsymbol{E}_{nt}}{\partial z} \cdot \mathrm{d}\boldsymbol{S} \exp\left(-\mathrm{i} \int^z k_n(z')\,\mathrm{d}z' \right)
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
&\frac{\partial A_m^+}{\partial z} \exp\left(+\mathrm{i} \int^z k_m(z')\,\mathrm{d}z' \right) - \frac{\partial A_m^-}{\partial z} \exp\left(-\mathrm{i} \int^z k_m(z')\,\mathrm{d}z' \right) \\
&\qquad = \sum_n A_n^+ \int \boldsymbol{E}_{mt} \times \frac{\partial \boldsymbol{H}_{nt}}{\partial z} \cdot \mathrm{d}\boldsymbol{S} \exp\left(+\mathrm{i} \int^z k_n(z')\,\mathrm{d}z' \right) \\
&\qquad\quad - \sum_n A_n^- \int \boldsymbol{E}_{mt} \times \frac{\partial \boldsymbol{H}_{nt}}{\partial z} \cdot \mathrm{d}\boldsymbol{S} \exp\left(-\mathrm{i} \int^z k_n(z')\,\mathrm{d}z' \right).
\end{aligned} \tag{21}$$

Adding and subtracting equations (20) and (21) we have

$$\frac{\partial A_m^\pm}{\partial z} = \sum_n C_{mn}^\pm e_{n\mp m} A_n^+ + \sum_n C_{mn}^\mp e_{-n\mp m} A_n^- \tag{22}$$

where

$$C_{mn}^\pm = \frac{1}{2} \int \left(\pm \boldsymbol{E}_{mt} \times \frac{\partial \boldsymbol{H}_{nt}}{\partial z} - \boldsymbol{H}_{mt} \times \frac{\partial \boldsymbol{E}_{nt}}{\partial z} \right) \cdot \mathrm{d}\boldsymbol{S} \tag{23}$$

and

$$e_{n\pm m} = \exp\left(\mathrm{i} \int^z (k_n(z') \pm k_m(z'))\,\mathrm{d}z' \right). \tag{24}$$

These equations, together with initial conditions on the $A_n^\pm(z)$ constitute an exact, formal solution to the problem.

To integrate them exactly is difficult since the right-hand side contains the unknowns $A_n^\pm(z)$. We can integrate them approximately, however, by observing that the right-hand side contains terms $\partial \boldsymbol{E}_{nt}/\partial z$ or $\partial \boldsymbol{H}_{nt}/\partial z$. Since $\boldsymbol{E}_{nt}$ and $\boldsymbol{H}_{nt}$ depend on z only because the guide is changing, and since this change is slow, their derivatives will be small. We deduce that $\partial A_m^\pm/\partial z$ are small and that over a period of several wavelengths the $A_m^\pm(z)$ do not change much from their original values. Consequently we can put $A_n^\pm(z) = A_n^\pm(0)$

on the right-hand sides of (22) and (24) which then become completely determinate allowing an approximate integration.

In many cases this final integration may be difficult to perform analytically so that a numerical approach or further approximation must follow. It is to be emphasized that the adiabatic approximation consists in assuming that the perturbation is slow so that the mode conversion is slight and that this approximation alone will always give a solution in the closed form of a definite integral.

Equations (22)–(24) are equivalent to the coupled mode equations derived by Snyder (1970, (16)–(18)) and used by Marcuse (1973).

3. The bent guide

3.1. Quasimode formalism

Let us choose an axis which follows the bend of the guide and let z be the distance of any point on the axis measured along the curve. If we define a set of mutually perpendicular unit vectors so that $\boldsymbol{\varepsilon}_3(z)$ points along the axis at z, $\boldsymbol{\varepsilon}_1(z)$ is in its plane of curvature, perpendicular to $\boldsymbol{\varepsilon}_3(z)$ and its sense defined by some convention and $\boldsymbol{\varepsilon}_2(z) = \boldsymbol{\varepsilon}_3(z) \times \boldsymbol{\varepsilon}_1(z)$ then

$$\boldsymbol{r} = x\boldsymbol{\varepsilon}_1(z) + y\boldsymbol{\varepsilon}_2(z) + \int^z \boldsymbol{\varepsilon}_3(z')\,\mathrm{d}z' \tag{25}$$

defines any point in space (figure 3).

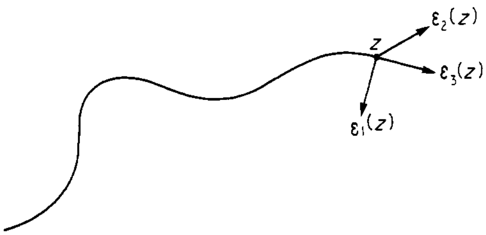

Figure 3. Local axes in a bending guide.

At any point z along the guide we construct an imaginary tangential straight guide as described and solve the usual wave equations (9) for its eigenmodes. Because our real guide has uniform cross section the solutions $\boldsymbol{E}_n^{\pm}(x, y)$ and $\boldsymbol{H}_n^{\pm}(x, y)$ will be independent of z, except insofar as the coordinates x and y refer to the local axes $\boldsymbol{\varepsilon}_1(z)$ and $\boldsymbol{\varepsilon}_2(z)$. The transverse components satisfy the orthonormality relations (13*a*) and the axial components are again given by (10) and (11).

We now define quasimodes of the bent guide by

$$\begin{pmatrix} \hat{\boldsymbol{E}}_n^{\pm}(\boldsymbol{r}, t) \\ \hat{\boldsymbol{H}}_n^{\pm}(\boldsymbol{r}, t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{E}_n^{\pm}(x, y) \\ \boldsymbol{H}_n^{\pm}(x, y) \end{pmatrix} \exp\{\mathrm{i}(\pm k_n z - \omega t)\}. \tag{26}$$

They satisfy: (i) the boundary conditions along the bent guide; (ii) Maxwell's equations provided we replace $\boldsymbol{\nabla}$ by

$$\boldsymbol{\nabla}_{\text{quasi}} \equiv \boldsymbol{\varepsilon}_1(z)\frac{\partial}{\partial x} + \boldsymbol{\varepsilon}_2(z)\frac{\partial}{\partial y} + \boldsymbol{\varepsilon}_3(z)\frac{\partial_{\text{quasi}}}{\partial z} \tag{27}$$

where the operator $\partial_{\text{quasi}}/\partial z$ ignores the z variation of the axes $\boldsymbol{\varepsilon}(z)$. Thus the $\boldsymbol{E}_n^{\pm}(x, y)$, $\boldsymbol{H}_n^{\pm}(x, y)$ satisfy equations similar to (5)–(8):

$$(\boldsymbol{\nabla}_{\mathrm{t}} \pm \mathrm{i}k_n\boldsymbol{\varepsilon}_3(z)) \cdot \epsilon \boldsymbol{E}_n^{\pm}(x, y) = 0 \tag{28}$$

$$(\boldsymbol{\nabla}_{\mathrm{t}} \pm \mathrm{i}k_n\boldsymbol{\varepsilon}_3(z)) \cdot \mu \boldsymbol{H}_n^{\pm}(x, y) = 0 \tag{29}$$

$$(\boldsymbol{\nabla}_{\mathrm{t}} \pm \mathrm{i}k_n\boldsymbol{\varepsilon}_3(z)) \times \boldsymbol{E}_n^{\pm}(x, y) = \frac{\mathrm{i}\omega\mu}{c}\boldsymbol{H}_n^{\pm}(x, y) \tag{30}$$

$$(\boldsymbol{\nabla}_{\mathrm{t}} \pm \mathrm{i}k_n\boldsymbol{\varepsilon}_3(z)) \times \boldsymbol{H}_n^{\pm}(x, y) = -\frac{\mathrm{i}\omega\epsilon}{c}\boldsymbol{E}_n^{\pm}(x, y). \tag{31}$$

For gentle bends the z dependence of the $\boldsymbol{E}_n^{\pm}$ and $\boldsymbol{H}_n^{\pm}$ is slow on the scale of a wavelength so that the quasimodes will 'nearly' satisfy Maxwell's equations.

It is necessary to consider the form of Maxwell's equations in a fixed coordinate system, congruent to the local system at z, but accounting for the changing local axes. We need to express the true operators, grad, div and curl, in terms of $\boldsymbol{\nabla}_{\text{quasi}}$ which ignores the z dependence of the axes. The results (see appendix 3) are

$$\begin{aligned}
\operatorname{grad} \phi &= \boldsymbol{\nabla}_{\text{quasi}}\phi - \frac{\sigma}{1+\sigma}\boldsymbol{\varepsilon}_3(z)\frac{\partial_{\text{quasi}}\phi}{\partial z} \\
\operatorname{div} \boldsymbol{V} &= \boldsymbol{\nabla}_{\text{quasi}} \cdot \boldsymbol{V} + \frac{1}{1+\sigma}\boldsymbol{V}_{\mathrm{t}} \cdot \boldsymbol{\nabla}_{\mathrm{t}}\sigma - \frac{\sigma}{1+\sigma}\frac{\partial_{\text{quasi}} V_z}{\partial z} \\
\operatorname{curl} \boldsymbol{V} &= \boldsymbol{\nabla}_{\text{quasi}} \times \boldsymbol{V} - \boldsymbol{\varepsilon}_3 \times \left(\frac{\sigma}{1+\sigma}\frac{\partial_{\text{quasi}}\boldsymbol{V}_{\mathrm{t}}}{\partial z} + \frac{V_z}{1+\sigma}\boldsymbol{\nabla}_{\mathrm{t}}\sigma\right)
\end{aligned} \tag{32}$$

where $\sigma = -x/\rho$ if we choose the x axis pointing towards the centre of curvature of the bend; ρ is the radius of curvature of the bend.

We now make our expansion for the fields

$$\begin{pmatrix} \boldsymbol{E}(\boldsymbol{r}, t) \\ \boldsymbol{H}(\boldsymbol{r}, t) \end{pmatrix} = \sum_n A_n^+(z)\begin{pmatrix} \hat{\boldsymbol{E}}_n^+(\boldsymbol{r}, t) \\ \hat{\boldsymbol{H}}_n^+(\boldsymbol{r}, t) \end{pmatrix} + \sum_n A_n^-(z)\begin{pmatrix} \hat{\boldsymbol{E}}_n^-(\boldsymbol{r}, t) \\ \hat{\boldsymbol{H}}_n^-(\boldsymbol{r}, t) \end{pmatrix} \tag{33}$$

of the guide in terms of the quasimodes; the arguments for the legitimacy of this expansion are as for the imperfect straight guide in § 2.2. To determine the mode coefficients we require that the fields satisfy Maxwell's equations (see (1)–(4)) using expression (32) for the vector operators. Substituting the field expansions (33) in the curl equations (3) and (4) we find after some manipulation that

$$\sum_n \left(\frac{\partial A_n^+}{\partial z} - \sigma\mathrm{i}k_nA_n^+\right)\begin{pmatrix} \hat{\boldsymbol{E}}_{nt}^+ \\ \hat{\boldsymbol{H}}_{nt}^+ \end{pmatrix} - A_n^+\boldsymbol{\nabla}_{\mathrm{t}}\sigma\begin{pmatrix} \hat{E}_{nz}^+ \\ \hat{H}_{nz}^+ \end{pmatrix} + \left(\frac{\partial A_n^-}{\partial z} + \sigma\mathrm{i}k_nA_n^-\right)\begin{pmatrix} \hat{\boldsymbol{E}}_{nt}^- \\ \hat{\boldsymbol{H}}_{nt}^- \end{pmatrix} - A_n^-\boldsymbol{\nabla}_{\mathrm{t}}\sigma\begin{pmatrix} \hat{E}_{nz}^- \\ \hat{H}_{nz}^- \end{pmatrix} = 0. \tag{34}$$

Similarly, substituting in the div equations (1) and (2) we find

$$\sum_n \left(\frac{\partial A_n^+}{\partial z} - \sigma\mathrm{i}k_nA_n^+\right)\begin{pmatrix} \hat{E}_{nz}^+ \\ \hat{H}_{nz}^+ \end{pmatrix} + A_n^+\boldsymbol{\nabla}_{\mathrm{t}}\sigma \cdot \begin{pmatrix} \hat{\boldsymbol{E}}_{nt}^+ \\ \hat{\boldsymbol{H}}_{nt}^+ \end{pmatrix} + \left(\frac{\partial A_n^-}{\partial z} + \sigma\mathrm{i}k_nA_n^-\right)\begin{pmatrix} \hat{E}_{nz}^- \\ \hat{H}_{nz}^- \end{pmatrix} + A_n^-\boldsymbol{\nabla}_{\mathrm{t}}\sigma \cdot \begin{pmatrix} \hat{\boldsymbol{E}}_{nt} \\ \hat{\boldsymbol{H}}_{nt} \end{pmatrix} = 0. \tag{35}$$

We can show that equations (35) derive from (34) by forming the curl of (34), considering the ε_3 component and using relations (28)–(31).

Equations (34) determine the coefficients $A_m^{\pm}(z)$ uniquely. They are more conveniently written using the convention (16).

We can extract the quantities $\partial A_m^{\pm}/\partial z$ by using the orthonormality relations (13*a*) to find

$$\frac{\partial A_m^{\pm}}{\partial z} = \pm\sum_n D_{mn}^{\pm} e_{n\mp m} A_n^{+} \pm \sum_n D_{mn}^{\mp} e_{-n\mp m} A_n^{-} \tag{36}$$

where

$$D_{mn}^{\pm} = \frac{1}{2}\int \mathrm{d}\boldsymbol{S}\,.\,(\boldsymbol{Y}_n \times \boldsymbol{E}_{mt} \pm \boldsymbol{H}_{mt} \times \boldsymbol{X}_n) \tag{37}$$

with

$$\boldsymbol{X}_n = \sigma \mathrm{i} k_n \boldsymbol{E}_{nt} + E_{nz} \nabla_t \sigma$$

and

$$\boldsymbol{Y}_n = \sigma \mathrm{i} k_n \boldsymbol{H}_{nt} + H_{nz} \nabla_t \sigma \tag{38}$$

and also $e_{n-m} = \exp\{\mathrm{i}(k_n - k_m)z\}$, for instance.

To solve equations (36) approximately for a gently bending guide we use arguments similar to those used in section (2.3) to replace the $A_n^{\pm}(z)$ on the right-hand side by their original values $A_n^{\pm}(0)$. The solution may then be found in the closed form of a definite integral, although for a guide which bends in a complicated way this final integral will be difficult to perform.

Two general comments are in order.

The definition (25) does not have a unique inverse and some points in space can have more than one coordinate representation. This is not important when we ask for the energy distribution among the modes at a point z along the guide. However, when we specify the fields at any point in space we need a convention to associate just one set of coordinates, (x, y, z) with that point.

Mathematically speaking mode conversion is caused by the quantities $\nabla_t \sigma$ and $\sigma \mathrm{i} k_n$, for if these are zero then equations (36) are satisfactorily solved by $A_n^{\pm}(z)$ = constant. These terms do have some physical significance; $\nabla_t \sigma = -\varepsilon_1/\rho$ mixes the transverse fields with the longitudinal fields in (36) and allows for the fact that as the guide changes direction the transverse fields are clearly in part developed from the axial fields of the adjacent sections of guide and *vice versa*. $\sigma \mathrm{i} k_n = x\mathrm{i}k_n/\rho$ accounts for the differential phase change measured parallel to the axis and at a distance x away from the axis; an off-axis point on a plane wavefront travels a different distance from an axial point as the wave negotiates a bend.

3.2. Example

The waveguide consisting of two infinite parallel perfectly conducting plates provides an illustrative example of how the theory can be applied to the bent waveguide. The plates are separated by a distance $2d_0$ and the medium between them has permeability μ and dielectric constant ϵ. To define a bend in this waveguide we set up local coordinates as described in § 3.1. We assume electromagnetic radiation is propagating in the positive z direction and there is no variation in the y direction. For $z < 0$ the guide is straight and the plates are situated at $x = \pm d_0$. For $0 < z < L$ the guide has a radius of curvature ρ; because of the local coordinate system the plates remain at $x = \pm d_0$ and we use this bend to define the positive x direction as directed towards the centre of curvature. For $L < z$ the guide is again straight.

To obtain a solution to any problem of mode conversion in the bend using (36) the eigenmodes of a straight guide of width $2d_0$ are required. These modes are given in appendix 4.

We look at the specific problem of unit electromagnetic energy in only the even TE mode $n = \alpha$ incident from $z < 0$ on the bend. We calculate the amplitudes of other modes which have energy transferred to them as a result of the bend. For this problem the boundary conditions to be applied to (36) are

$$A_{en}^{E+}(0) = \delta_{n\alpha}, \qquad A_{on}^{E+}(0) = A_{en}^{E-}(L) = A_{on}^{E-}(L) = 0$$
$$A_{en}^{M+}(0) = A_{on}^{M+}(0) = A_{en}^{M-}(L) = A_{on}^{M-}(L) = 0 \tag{39}$$

where the notation describing TE and TM even and odd modes is given in appendix 4. To facilitate straightforward solution of (36) we make the approximation of replacing all the coefficients A by their values at $z = 0$. This will be valid as long as L is sufficiently small and ρ sufficiently large and also under these circumstances the quantities $A_{en}^{E-}(0)$, $A_{on}^{E-}(0)$, $A_{en}^{E-}(0)$ and $A_{on}^{E-}(0)$ will be of first order and may be neglected. The matrix elements in (36) are easily calculated and it is found that energy is transferred by this odd deviation only to the TE odd modes from the initial TE even mode by the bend. The matrix elements connecting TE even and odd modes are given by

$$D_{omen}^{\pm EE} = \frac{i(-1)^{m+n}}{\rho d_0} \frac{\gamma_{om}\gamma_{en}}{(\gamma_{on}^2 - \gamma_{en}^2)^2} \frac{(k_{om}^E \pm k_{en}^E)^2}{(k_{om}^E k_{en}^E)^{1/2}}. \tag{40}$$

Since the matrix elements are independent of z and the $A_{e\alpha}^{E+}(z)$ is assumed to be constant (36) is now particularly easy to integrate to obtain the mode ampitudes. The result is

$$A_{om}^{E+}(z) \simeq \frac{D_{ome\alpha}^{+EE}}{i(k_{e\alpha}^E - k_{om}^E)}[\exp\{i(k_{e\alpha}^E - k_{om}^E)z\} - 1]$$
$$A_{e\alpha}^{E+}(z) \simeq 1$$
$$A_{om}^{E-}(z) \simeq -\frac{D_{ome\alpha}^{-EE}}{i(k_{e\alpha}^E + k_{om}^E)}[\exp\{i(k_{e\alpha}^E + k_{om}^E)z\} - \exp\{i(k_{e\alpha}^E + k_{om}^E)L\}] \tag{41}$$

where the coefficients A_{om}^{E+} refer to the mth odd TE modes and all other coefficients remain zero to first order.

Assuming the coefficients remain small, which is the basic requirement for (41) to be valid, it is clear that the coefficients are oscillatory functions and do not steadily grow as the radiation negotiates the bend. A gentle bend does not have a cumulative effect and the coefficients are not proportional to the length of bend traversed. To gain some insight into when the coefficients do remain small we consider the particular case of mode conversion between two low order modes (α, m small integers) and a radiation frequency such that these modes are far from cut off. In this situation the coefficients $A_{om}^{E+}(z)$ have an amplitude of oscillation given approximately by $(2d/\rho)(d/\lambda)^2$ where λ is the free-space wavelength of the radiation. The amplitude is of the form we might expect on physical grounds and in particular demonstrates that a radius of bend much larger than the guide width is required for (41) to be valid.

Future work will deal with bends in round dielectric guides or optical fibres but it is expected the solutions will retain the same general form.

4. Conclusion

We have expressed the fields of an imperfect waveguide in terms of a complete set of fields which are adapted to the local form of the guide and whose phase is that acquired at previous points along the guide. As a result we have been able to calculate the mode conversion caused by local changes along an essentially straight guide or by any bends it suffers.

We have seen that provided the changes are slow or the bends are gentle the modes of the guide will themselves distort and power will convert only weakly between them. In this case it is thought that a low-order perturbation treatment would over-estimate the mode conversion.

It is interesting to note that the exact equations (22) and (36) can be written formally as

$$\frac{\partial A(z)}{\partial z} = \mathbf{M}(z)A(z)$$

where $A(z)$ is a vector whose elements are the quantities $A_n^{\pm}(z)$ and $\mathbf{M}^{\pm}(z)$ represents the coupling matrix. This equation has the formal solution

$$A(z) = \exp\left(\int_0^z \mathbf{M}(z')\,\mathrm{d}z'\right)A(0)$$

where the exponential is to be interpreted as its power series expansion.

Our 'adiabatic approximation' which consists in replacing the quantities $A_n^{\pm}(z)$ by $A_n^{\pm}(0)$ on the right of equations (22) and (36) is equivalent to truncating the expansion for the exponential after the second term. The later terms will account for the multiple scattering of power between modes.

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Appendix 1. Derivation of the wave equations for quasimodes

By separating the fields into their axial and transverse components:

$$\boldsymbol{E}_n^{\pm} = \boldsymbol{E}_{nt}^{\pm} + \boldsymbol{\varepsilon}_3 E_{nz}^{\pm}, \qquad \boldsymbol{H}_n^{\pm} = \boldsymbol{H}_{nt}^{\pm} + \boldsymbol{\varepsilon}_3 H_{nz}^{\pm}$$

we may write Maxwell's equations ((5)–(8) or (28)–(31)) for the quasimodes as

$$\nabla_t \cdot \epsilon \boldsymbol{E}_{nt}^{\pm} \pm \mathrm{i}k_n \epsilon E_{nz}^{\pm} = 0 \qquad \text{(A1.1)}$$

$$\nabla_t \cdot \mu \boldsymbol{H}_{nt}^{\pm} \pm \mathrm{i}k_n \mu H_{nz}^{\pm} = 0 \qquad \text{(A1.2)}$$

$$\nabla_t \times \boldsymbol{E}_{nt}^{\pm} = \frac{i\omega\mu}{c}\boldsymbol{\varepsilon}_3 H_{nz}^{\pm} \tag{A1.3}$$

$$\boldsymbol{\varepsilon}_3 \times \nabla_t E_{nz}^{\pm} \mp ik_n \boldsymbol{\varepsilon}_3 \times \boldsymbol{E}_{nt}^{\pm} = -\frac{i\omega\mu}{c}\boldsymbol{H}_{nt}^{\pm} \tag{A1.4}$$

$$\nabla_t \times \boldsymbol{H}_{nt}^{\pm} = -\frac{i\omega\epsilon}{c}\boldsymbol{\varepsilon}_3 E_{nz}^{\pm} \tag{A1.5}$$

$$\boldsymbol{\varepsilon}_3 \times \nabla_t H_{nz}^{\pm} \mp ik_n \boldsymbol{\varepsilon}_3 \times \boldsymbol{H}_{nt}^{\pm} = \frac{i\omega\epsilon}{c}\boldsymbol{E}_{nt}^{\pm}. \tag{A1.6}$$

Operating on equation (A1.3) with $\nabla_t \times (1/\mu)$ we find

$$\nabla_t \times \frac{1}{\mu}\nabla_t \times \boldsymbol{E}_{nt}^{\pm} = -\frac{i\omega}{c}\boldsymbol{\varepsilon}_3 \times \nabla_t H_{nz}^{\pm}$$

and substituting the right-hand side of this equation into (A1.6) it follows that

$$\nabla_t \times \frac{1}{\mu}\nabla_t \times \boldsymbol{E}_{nt}^{\pm} - \frac{\omega^2\epsilon}{c^2}\boldsymbol{E}_{nt}^{\pm} = \pm\frac{\omega k_n}{c}\boldsymbol{\varepsilon}_3 \times \boldsymbol{H}_{nt}^{\pm}. \tag{A1.7}$$

Operating with $(1/\mu)\nabla_t(1/\epsilon)$ (A1.1) gives

$$\frac{1}{\mu}\nabla_t\left(\frac{1}{\epsilon}\nabla_t \cdot \epsilon\boldsymbol{E}_{nt}^{\pm}\right) = \mp\frac{ik_n}{\mu}\nabla_t E_{nz}^{\pm}$$

and subtracting this from (A1.7) gives

$$\nabla_t \times \frac{1}{\mu}\nabla_t \times \boldsymbol{E}_{nt}^{\pm} - \frac{1}{\mu}\nabla_t\left(\frac{1}{\epsilon}\nabla_t \cdot \epsilon\boldsymbol{E}_{nt}^{\pm}\right) - \frac{\omega^2\epsilon}{c^2}\boldsymbol{E}_{nt}^{\pm} = \pm\frac{\omega k_n}{c}\boldsymbol{\varepsilon}_3 \times \boldsymbol{H}_{nt}^{\pm} \pm \frac{ik_n}{\mu}\nabla_t E_{nz}^{\pm}. \tag{A1.8}$$

Finally, operating on (A1.4) with $\mp ik_n\boldsymbol{\varepsilon}_3/\mu$ gives

$$\frac{k_n^2}{\mu}\boldsymbol{E}_{nt}^{\pm} \pm \frac{ik_n}{\mu}\nabla_t E_{nz}^{\pm} = \mp\frac{\omega k_n}{c}\boldsymbol{\varepsilon}_3 \times \boldsymbol{H}_{nt}^{\pm}$$

and on eliminating $\boldsymbol{H}_{nt}^{\pm}$ from this and equation (A1.8) it follows that

$$\mu\nabla_t \times \frac{1}{\mu}\nabla_t \times \boldsymbol{E}_{nt}^{\pm} - \nabla_t\left(\frac{1}{\epsilon}\nabla_t \cdot \epsilon\boldsymbol{E}_{nt}^{\pm}\right) - \left(\frac{\omega^2\mu\epsilon}{c^2} - k_n^2\right)\boldsymbol{E}_{nt}^{\pm} = 0 \tag{A1.9}$$

which is the desired wave equation.

Because Maxwell's equations are invariant under the transformation $\epsilon \to \mu$, $\mu \to \epsilon$, $\boldsymbol{E} \to \boldsymbol{H}$, $\boldsymbol{H} \to -\boldsymbol{E}$ the equation for $\boldsymbol{H}_{nt}^{\pm}$ is

$$\epsilon\nabla_t \times \frac{1}{\epsilon}\nabla_t \times \boldsymbol{H}_{nt}^{\pm} - \nabla_t\left(\frac{1}{\mu}\nabla_t \cdot \mu\boldsymbol{H}_{nt}^{\pm}\right) - \left(\frac{\omega^2\mu\epsilon}{c^2} - k_n^2\right)\boldsymbol{H}_{nt}^{\pm} = 0. \tag{A1.10}$$

Appendix 2. Orthogonality of the quasimodes

Consider two distinct modes labelled by n, m. For the waves travelling in the ε_3 direction we have

$$(\nabla_t + ik_n\varepsilon_3) \times E_n^+ = \frac{i\omega\mu}{c} H_n^+$$

$$(\nabla_t + ik_n\varepsilon_3) \times E_m^+ = \frac{i\omega\mu}{c} H_m^+ .$$

Scalar multiplying the above equations by H_m^+ and H_n^+ respectively and subtracting gives

$$H_m^+ \cdot (\nabla_t + ik_n\varepsilon_3) \times E_n^+ - H_n^+ \cdot (\nabla_t + ik_m\varepsilon_3) \times E_m^+ = 0. \tag{A2.1}$$

Similar consideration of the curl equations for H_n^+ and H_m^+ gives

$$E_m^+ \cdot (\nabla_t + ik_n\varepsilon_3) \times H_n^+ - E_n^+ \cdot (\nabla_t + ik_m\varepsilon_3) \times H_m^+ = 0. \tag{A2.2}$$

Adding equations (1) and (2) we find, after some manipulation

$$\nabla_t \cdot (E_n^+ \times H_m^+ - E_m^+ \times H_n^+) + i(k_n + k_m)\varepsilon_3 \cdot (E_n^+ \times H_m^+ - E_m^+ \times H_n^+) = 0.$$

If we integrate the axial component of this expression over a surface perpendicular to the (local) direction of the guide then we can use the two dimensional form of the divergence to obtain

$$\oint_C n \cdot (E_n^+ \times H_m^+ - E_m^+ \times H_n^+)\, dl + i(k_n + k_m) \int (E_{nt}^+ \times H_{mt}^+ - E_{mt}^+ \times H_{nt}^+) \cdot dS = 0.$$

The line integral with respect to l is taken round the boundary C of the surface S; $\boldsymbol{n}$ is the outward normal of this boundary. In deriving this expression we have noted that the axial components of the field disappear on taking the axial components of the surface integrand.

In a waveguide bounded by a conductor we can rearrange the triple products in the line integral to involve terms in $\boldsymbol{n} \times \boldsymbol{E}$ so that taking the conductor as the boundary the line integral vanishes. In a dielectric guide we can let the boundary extend to infinity where the line integral vanishes if it involves a guided mode. For radiation modes we can impose periodic boundary conditions. In any case we are left with the condition

$$(k_n + k_m) \int (E_{nt}^+ \times H_{mt}^+ - E_{mt}^+ \times H_{nt}^+) \cdot dS = 0. \tag{A2.3}$$

Now if we look instead at the mode with eigenvalue $-k_m$ travelling in the $-\varepsilon_3$ direction and use our convention

$$E_{mt}^+ = E_{mt}^- = E_{mt}$$

$$H_{mt}^+ = -H_{mt}^- = H_{mt}.$$

Then (A2.3) gives

$$(k_n + k_m) \int (E_{nt} \times H_{mt} - E_{mt} \times H_{nt}) \cdot dS = 0$$

$$(k_n - k_m) \int (E_{nt} \times H_{mt} + E_{mt} \times H_{nt}) \cdot dS = 0$$

from which we deduce that

$$\int (\boldsymbol{H}_{mt} \times \boldsymbol{E}_{nt}) \cdot \mathrm{d}\boldsymbol{S} = 0 \qquad m \neq n. \tag{A2.4}$$

Appendix 3. Differential operators in the coordinate system of the bent guide

The displacement from a point $\mathbf{r}$, specified by coordinates x, y, z to a neighbouring point defined by increments $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ is given by

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}x\boldsymbol{\varepsilon}_1(z) + \mathrm{d}y\boldsymbol{\varepsilon}_2(z) + \mathrm{d}z\boldsymbol{\varepsilon}_3(z) + \left(x\frac{\mathrm{d}\boldsymbol{\varepsilon}_1}{\mathrm{d}z} + y\frac{\mathrm{d}\boldsymbol{\varepsilon}_2}{\mathrm{d}z}\right)\mathrm{d}z. \tag{A3.1}$$

If only x or y are varied then the displacements are $\mathrm{d}x$, $\mathrm{d}y$ along the local $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ axes respectively, so that the transverse component of grad is

$$\mathrm{grad}_\mathrm{t} = \boldsymbol{\varepsilon}_1(z)\frac{\partial}{\partial x} + \boldsymbol{\varepsilon}_2(z)\frac{\partial}{\partial y}. \tag{A3.2}$$

If only z is varied, however, the total displacement is given by

$$(\mathrm{d}\boldsymbol{r})_z = \left(\boldsymbol{\varepsilon}_3(z) + x\frac{\mathrm{d}\boldsymbol{\varepsilon}_1}{\mathrm{d}z} + y\frac{\mathrm{d}\boldsymbol{\varepsilon}_2}{\mathrm{d}z}\right)\mathrm{d}z. \tag{A3.3}$$

Because of the way we have chosen $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ it follows that $\mathrm{d}\boldsymbol{\varepsilon}_2/\mathrm{d}z = 0$ and $\mathrm{d}\boldsymbol{\varepsilon}_1/\mathrm{d}z$ is parallel or antiparallel to $\boldsymbol{\varepsilon}_3(z)$. Hence the increment $(\mathrm{d}\boldsymbol{r})_z$ is also in the direction $\boldsymbol{\varepsilon}_3(z)$ but of magnitude $(1 + x\boldsymbol{\varepsilon}_3(z) \cdot \mathrm{d}\boldsymbol{\varepsilon}_1/\mathrm{d}z)\,\mathrm{d}z$.

The axial component of grad is thus equal to

$$\frac{\boldsymbol{\varepsilon}_3(z)}{1+\sigma}\frac{\partial_{\mathrm{quasi}}}{\partial z} \qquad \text{where} \qquad \sigma = x\boldsymbol{\varepsilon}_3(z) \cdot \frac{\mathrm{d}\boldsymbol{\varepsilon}_1}{\mathrm{d}z} = -\frac{x}{\rho} \tag{A3.4}$$

if we choose $\boldsymbol{\varepsilon}_1(z)$ pointing towards the centre of curvature.

Thus

$$\mathrm{grad}\,\phi = \boldsymbol{\nabla}_{\mathrm{quasi}}\phi - \frac{\sigma}{1+\sigma}\boldsymbol{\varepsilon}_3(z)\frac{\partial_{\mathrm{quasi}}}{\partial z}\phi. \tag{A3.5}$$

The expressions for the operators div and curl in our orthogonal curvilinear coordinate system then follow (see Morse and Feshbach 1953, pp 21–31) as

$$\mathrm{div}\,\boldsymbol{V} = \boldsymbol{\nabla}_{\mathrm{quasi}} \cdot \boldsymbol{V} - \frac{\sigma}{1+\sigma}\frac{\partial V_z}{\partial z} + \frac{\boldsymbol{V}_\mathrm{t} \cdot \boldsymbol{\nabla}_\mathrm{t}\sigma}{1+\sigma} \tag{A3.6}$$

$$\mathrm{curl}\,\boldsymbol{V} = \boldsymbol{\nabla}_{\mathrm{quasi}} \times \boldsymbol{V} - \boldsymbol{\varepsilon}_3 \times \left(\frac{\sigma}{1+\sigma}\frac{\partial \boldsymbol{V}_\mathrm{t}}{\partial z} + \frac{V_z}{1+\sigma}\boldsymbol{\nabla}_\mathrm{t}\sigma\right). \tag{A3.7}$$

Appendix 4. Modes of a parallel plate waveguide

The modes of a waveguide consisting of two infinite parallel perfectly conducting planes with a plane separation $2d$ are well known. We take the planes to be at $x = \pm d$ and assume the radiation is propagating in the $+z$ direction. With the coordinate system so defined the modes fall into groups of TE and TM modes which may be further classified as even or odd. Each of these sets has modes corresponding to forward and backward waves. To distinguish the different cases subscripts e and o denote even and odd; superscripts E and M for TE and TM; and the subscripts n and m are quantum numbers. The modes for waves travelling forward are given; the dependence e^{ikz} is omitted. The negative z direction waves are obtained using (16); the normalization used is (13a).

A.4.1. TE *even modes*

$$\begin{aligned}
H_{nx} &= -\frac{iB_e k^E_{en}}{\mu\gamma_{en}}\sin\gamma_{en}x & E_{nx} &= 0 \\
H_{ny} &= 0 & E_{ny} &= i\omega B_e \sin\gamma_{en}x \\
H_{nz} &= \frac{B_e}{\mu}\cos\gamma_{en}x & E_{nz} &= 0 \\
\gamma_{en} &= \frac{n\pi}{d} \quad n = 1, 2, 3, \ldots & B_e^2 &= \frac{c\mu\gamma^2_{en}}{\omega k^E_{en} d}.
\end{aligned} \tag{A4.1}$$

A.4.2. TE *odd modes*

$$\begin{aligned}
H_{nx} &= \frac{ik^E_{on}B_o}{\mu\gamma_{on}}\cos\gamma_{on}x & E_{nx} &= 0 \\
H_{ny} &= 0 & E_{ny} &= -\frac{i\omega B_o}{c\gamma_{on}}\cos\gamma_{on}x \\
H_{nz} &= \frac{B_o}{\mu}\sin\gamma_{on}x & E_{nz} &= 0 \\
\gamma_{on} &= (n-\tfrac{1}{2})\frac{\pi}{d} & B_0^2 &= \frac{c\mu\gamma^2_{on}}{\omega k^E_{on} d}
\end{aligned} \tag{A4.2}$$

A.4.3. TM *even modes*

$$\begin{aligned}
H_{nx} &= 0 & E_{nx} &= -\frac{ik^M_{en}D_{en}}{\delta_{en}}\sin\delta_{en}x \\
H_{ny} &= -\frac{i\epsilon\omega}{c\delta_{en}}D_{en}\sin\delta_{en}x & E_{ny} &= 0 \\
H_{nz} &= 0 & E_{nz} &= D_{en}\cos\delta_{en}x \\
\delta_{en} &= (n-\tfrac{1}{2})\frac{\pi}{d} \quad n = 1, 2, 3, \ldots & D^2_{en} &= \frac{\delta^2_{en}c}{\epsilon\omega k^M_{en} d}
\end{aligned} \tag{A4.3}$$

A.4.4 TM odd modes

$$
\begin{aligned}
H_{nx} &= 0 & E_{nx} &= \mathrm{i}k^{\mathrm{M}}_{\mathrm{o}n}D_{\mathrm{o}n}\cos\delta_{\mathrm{o}n}x \\
H_{ny} &= \frac{\mathrm{i}\epsilon\omega D_{\mathrm{o}n}}{\delta_{\mathrm{o}n}c}\cos\delta_{\mathrm{o}n}x & E_{ny} &= 0 \\
H_{nz} &= 0 & E_{nz} &= D_{\mathrm{o}n}\sin\delta_{\mathrm{o}n}x \\
\delta_{\mathrm{o}n} &= \frac{n\pi}{d} \quad n = 1, 2, 3, \ldots & D^2_{\mathrm{o}n} &= \frac{c\delta^2_{\mathrm{o}n}}{\epsilon\omega k^{\mathrm{M}}_{\mathrm{o}n}d}.
\end{aligned}
\tag{A4.4}
$$

Throughout

$$
\begin{aligned}
k^{\mathrm{E}} &= \left(\frac{\mu\epsilon\omega^2}{c^2} - \gamma^2\right)^{1/2} \\
k^{\mathrm{M}} &= \left(\frac{\mu\epsilon\omega^2}{c^2} - \delta^2\right)^{1/2}.
\end{aligned}
\tag{A4.5}
$$

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